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1. (a) Complete the statement of the formula for integration by parts.

$$\int u\,dv \quad = \quad \underline{\quad uv \quad - \quad \int v\,du \quad}$$

(b) Our proof in class of the integration by parts formula relied on two important results from Calculus I and II.
Name these results!

Calculus I result: <u>Product Rule for derivatives</u>

Calculus II result: <u>Fundamental Theorem of Calculus</u>

(c) We may evaluate the integral

$$\int 2x\ln(x+2)\,dx$$

using integration by parts with $v = x^2 - 4$.
This is an unusual choice for v, but it will work!
Fill in the three remaining blanks.

$$u \quad = \quad \underline{\ln(x+2)} \qquad\qquad dv \quad = \quad \underline{2x\,dx}$$

$$du \quad = \quad \underline{\frac{1}{x+2}\,dx} \qquad\qquad v \quad = \quad \underline{x^2-4}$$

(d) Evaluate the integral in (c) using the scheme above.

$$\int 2x\ln(x+2)\,dx \quad = \quad \int u\,dv \quad = \quad uv \quad - \quad \int v\,du$$

$$= \quad (x^2-4)\ln(x+2) \quad - \quad \int (x^2-4)\cdot\left(\frac{1}{x+2}\right)\,dx$$

$$= \quad (x^2-4)\ln(x+2) \quad - \quad \int (x-2)\ dx$$

$$\underline{= \quad (x^2-4)\ln(x+2) \quad - \quad \frac{x^2}{2} \quad + \quad 2x \quad + \quad C}$$

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2. Here are three labeled integrals.

$$(A)\quad \int_0^1 \frac{1}{\sqrt{x}}\,dx \qquad (B)\quad \int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad (C)\quad \int_0^2 \frac{1}{\sqrt{x+2}}\,dx$$

Fill in each blank with a capital letter, and circle the correct value.

(a) Integral __C__ is a proper integral.

(b) Integral __B__ is an improper integral that diverges.

(c) Integral __A__ is an improper integral that converges to ...

$-1/2$ $\quad 0 \quad$ $1/2$ $\quad 1 \quad$ $\boxed{2}$

$$(A)\quad \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\longrightarrow 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\longrightarrow 0^+} 2\sqrt{x}\big|_a^1 = \lim_{a\longrightarrow 0^+} 2\sqrt{1}-2\sqrt{a} \quad = \quad 2$$

$$(B)\quad \int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\longrightarrow\infty}\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\longrightarrow\infty} 2\sqrt{x}\big|_1^\infty = \lim_{b\longrightarrow\infty} 2\sqrt{b}-2\sqrt{1} = \infty$$

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3. (a) Fill in the blanks to complete the definition of the definite integral $\int_a^b f(x)\,dx$.
Let $f(x)$ be a continuous function on the interval $[a,b]$. Let the points $a = x_0, x_1, x_2, \ldots, x_n = b$ partition $[a,b]$ into n equal sub-intervals. The length of each sub-interval is

$$\Delta x = \underline{\quad \frac{b-a}{n} \quad}.$$

Let x_i^* be a sample point in the i-th subinterval. Then the definition of the definite integral is:

$$\int_a^b f(x)\,dx = \underline{\quad \lim_{n \longrightarrow \infty} \left(\sum_{i=1}^{n} f(x_i^*) \Delta x \right) \quad}$$

(b) A student is computing $\int_0^1 x^2\,dx$ using the above definition. She selects the right endpoints as the sample points, so that $x_i^* = x_i$.
Which Riemann sum arises at an intermediate step?

Circle a Roman numeral.

i. $R_n = \frac{1}{n^2}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$

ii. $R_n = \frac{1}{n^2}\left(1^3 + 2^3 + 3^3 + \cdots + n^3\right)$

iii. $R_n = \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$ $\checkmark$ See classnotes or text.

iv. $R_n = \frac{1}{n^3}\left(1^3 + 2^3 + 3^3 + \cdots + n^3\right)$

v. $R_n = \frac{1}{n}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$

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4. (a) Let $f(x)$ be a continuous function on the interval $[a, b]$.
Complete the statement of the Fundamental Theorem of Calculus.

(I) If $g(x) = \int_a^x f(t)\,dt$, then $\underline{\quad g'(x) = f(x) \quad}$

(II) If $F(x)$ is an antiderivative of $f(x)$, then $\underline{\quad \int_a^b f(x)\,dx = F(b) - F(a) \quad}$

(b) Suppose that

$$G(x) \quad = \quad \int_0^{\sqrt{x}} 8e^{-t^2}\,dt.$$

Then $G'(4)$ is of the form Ae^B.
Circle the values for A and B.

$A = 1$ $\boxed{A = 2}$ $A = 4$ $A = 8$ $A = 16$

$B = -16$ $\boxed{B = -4}$ $B = -2$ $B = 4$ $B = 16$

$$G'(x) = 8e^{-(\sqrt{x})^2} \cdot \frac{d}{dx}\left(\sqrt{x}\right) = 8e^{-x} \cdot \left(\frac{1}{2\sqrt{x}}\right).$$

$$G'(4) = 8e^{-4} \cdot \frac{1}{4} = 2e^{-4}$$

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5. Each of the following integrals can be evaluated with a u-substitution.
   Circle the BEST substitution.
   You are not being asked to evaluate these integrals!

   (a)

$$\int \frac{e^{3t}-e^{-3t}}{e^{3t}+e^{-3t}}\,dt \quad = \quad \frac{1}{3}\int \frac{du}{u} \quad = \quad \frac{1}{3}\ln(u)+C \quad = \cdots$$

   i. $u = 3t$
   ii. $u = e^{3t}$
   iii. $u = e^{-3t}$
   iv. $u = e^{3t} - e^{-3t}$
   v. $\boxed{u = e^{3t} + e^{-3t}}$ $du = 3(e^{3t} - e^{-3t})\,dt$

   (b)

$$\int \frac{\cos(x)}{1+\sin^2(x)}\,dx \quad = \quad \int \frac{1}{1+u^2}\,du \quad = \quad \arctan(u)+C \quad = \cdots$$

   i. $u = \cos(x)$
   ii. $\boxed{u = \sin(x)}$ $du = \cos(x)\,dx$
   iii. $u = \sin^2(x)$
   iv. $u = 1 + \sin^2(x)$
   v. $u = \sqrt{\sin(x)}$

   (c)

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx \quad = \quad \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du \quad = \quad \frac{1}{2}\arcsin(u)+C \quad = \quad \cdots$$

   i. $\boxed{u = 2x}$ $du = 2\,dx$
   ii. $u = 4x$
   iii. $u = 1 - 4x^2$
   iv. $u = \sqrt{1-4x^2}$
   v. $u = \frac{1}{\sqrt{1-4x^2}}$

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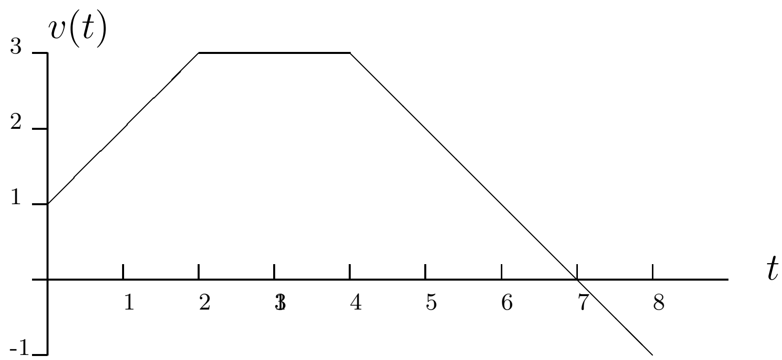
6. The graph shows the velocity $v(t)$ of a particle on a number line as a function of time from $t = 0$ to $t = 8$. The units are cm/sec.



(a) The value of the definite integral $\int_4^7 v(t)\,dt$ is

-3 0 3 $\boxed{4.5}$ 9 (area of a triangle)

(b) What is the **displacement** (in cm) of the particle between $t = 4$ and $t = 8$?

-4 -2 0 2 $\boxed{4}$ ($\int_4^8 v(t)\,dt = 4.5 - 0.5$)

(c) What is the **total distance** (in cm) traveled by the particle between $t = 4$ and $t = 8$?

1 2 3 4 $\boxed{5}$ ($\int_4^8 |v(t)|\,dt = 4.5 + 0.5$)

(d) At which of the following times is the particle farthest from its position at $t = 0$?

$t = 4$ $t = 5$ $t = 6$ $\boxed{t = 7}$ $t = 8$

(Particle starts moving back toward its initial position at $t = 7$.)

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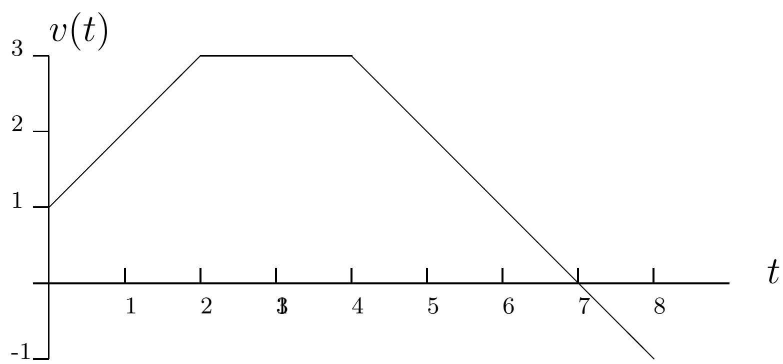
7. We continue discussing the function $v(t)$ from the previous problem.
The graph is reproduced here for your convenience.
We will use various methods to approximate

$$\int_0^8 v(t)\,dt$$



(a) The midpoint rule with $n = 4$ gives CIRCLE A LOWER CASE ROMAN NUMERAL

i. $M_4 = v(1) + v(3) + v(5) + v(7)$
ii. $\boxed{M_4 = 2\,(v(1) + v(3) + v(5) + v(7))}$
iii. $M_4 = v(1) + 2v(3) + 2v(5) + v(7)$
iv. $M_4 = v(2) + v(4) + v(6) + v(8)$
v. $M_4 = \frac{1}{2}\,(v(1) + v(3) + v(5) + v(7))$

(b) Simpson's rule with $n = 4$ gives GIVE AN EXPRESSION SIMILAR IN FORM TO THE CHOICES IN (A)

$$S_4 \quad = \quad \tfrac{2}{3}[v(0) + 4v(2) + 2v(4) + 4v(6) + v(8)]$$

(c) The trapezoidal rule with $n = 2$ gives

$T_2 = -3$ $T_2 = 3$ $T_2 = 6$ $T_2 = 8$ $\boxed{T_2 = 12}$

$$T_2 = \frac{4}{2}\,(v(0) + 2v(4) + v(8)) = 2(1 + 2 \cdot 3 - 1) = 12$$

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8. The differentiable function g satisfies

$$\int_0^4 g(x)\,dx = 16 \qquad \int_2^4 g(x)\,dx = 5 \qquad g(0) = 0 \qquad g(2) = 20 \qquad g'(2) = 3.$$

Also, $g(x)$ is an **odd** function.

Fill in each blank with the correct numerical value.

If the expression cannot be determined from the given information, write a question mark in the blank.

$\underline{\ -16\ }$ $\displaystyle\int_{-4}^0 g(x)\,dx \quad = -\int_0^4 g(x)\,dx = -16$ because g is odd.

$\underline{\ 11\ }$ $\displaystyle\int_0^2 g(x)\,dx \quad = \int_0^4 g(x)\,dx - \int_2^4 g(x)\,dx = 16-5 = 11$

$\underline{\ 8\ }$ $\displaystyle\int_0^2 x\cdot g(x^2)\,dx \quad = \frac{1}{2}\int_0^4 g(u)\,du = \frac{1}{2}\cdot 16 = 8$

$$\left\{\begin{array}{l} u = x^2 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \\ x = 0 \Longrightarrow u = 0 \\ x = 2 \Longrightarrow u = 4 \end{array}\right\}$$

$\underline{\ 20\ }$ $\displaystyle\int_0^2 g'(x)\,dx \quad = g(x)|_0^2 = g(20) - g(0) = 20-0 = 20$ by FTC

$\underline{\ -14\ }$ $\displaystyle\int_0^2 x\cdot g''(x)\,dx$ HINT: INTEGRATE BY PARTS $u = x; \quad dv = g''(x)dx$

$$\int_0^2 u\,dv = uv|_0^2 - \int_0^2 v\,du = x\cdot g'(x)|_0^2 - \int_0^2 g'(x)\,dx = 2\cdot 3 - 20 = -14$$

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9. The "Total Change Theorem" was discussed in class and the text.

(a) State the Total Change Theorem in words.

$\boxed{\text{The integral of a rate of change is the total change.}}$

(b) State the Total Change Theorem as a mathematical formula.

YOUR ANSWER SHOULD INVOLVE A DEFINITE INTEGRAL.

$\int_a^b F'(x)\,dx \quad = \quad F(b) - F(a)$

(c) A honeybee population starts with 100 bees at time $t = 0$ and increases at a rate of $n'(t)$ bees per week. Explain in a few words what the following expression represents.

$$100 \; + \; \int_0^{15} n'(t)\,dt$$

$\boxed{\text{The expression represents the number of bees after 15 weeks.}}$

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10. Fill in each blank line with Maple's output for the given commands.

(a) > convert($(3x^2 - 4x + 5)/(x-1)(x^2+1)$,parfrac,$x$);

$$\frac{2}{x-1} + \frac{x-3}{x^2+1}$$

(This was done in class; you could also use "expand" command on calculator—with care used for parentheses.)

(b) > int$((3x^2 - 4x + 5)/(x-1)(x^2+1), x)$;

$$\int \frac{3x^2 - 4x + 5}{(x-1)(x^2+1)}\, dx = \int \left(\frac{2}{x-1} + \frac{x-3}{x^2+1} \right) dx$$

$$= \quad 2\ln|x-1| \quad + \quad \frac{1}{2}\ln(x^2+1) \quad - \quad 3\arctan(x)$$

(Maple omits the "$+C$" term.)